

COCHARACTERS OF POLYNOMIAL IDENTITIES OF UPPER TRIANGULAR MATRICES

SILVIA BOUMOVA, VESSELIN DRENSKY

To Georgi Genov – teacher, colleague and friend,
on the occasion of his retirement

ABSTRACT. Let $T(U_k)$ be the T-ideal of the polynomial identities of the algebra of $k \times k$ upper triangular matrices over a field of characteristic zero. We give an easy algorithm which calculates the generating function of the cocharacter sequence $\chi_n(U_k) = \sum_{\lambda \vdash n} m_\lambda(U_k) \chi_\lambda$ of the T-ideal $T(U_k)$. Applying this algorithm we have found the explicit form of the multiplicities $m_\lambda(U_k)$ in two cases: (i) for the “largest” partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ which satisfy $\lambda_{k+1} + \dots + \lambda_n = k - 1$; (ii) for the first several k and any λ .

INTRODUCTION

We fix a field K of characteristic 0 and consider unital associative algebras over K only. For a background on PI-algebras and details of the results discussed in the introduction we refer to the book [D6]. Let R be a PI-algebra and let

$$T(R) \subset K\langle X \rangle = K\langle x_1, x_2, \dots \rangle$$

be the T-ideal of its polynomial identities, where $K\langle X \rangle$ is the free associative algebra of countable rank. One of the most important objects in the quantitative study of the polynomial identities of R is the cocharacter sequence

$$\chi_n(R) = \sum_{\lambda \vdash n} m_\lambda(R) \chi_\lambda, \quad n = 0, 1, 2, \dots,$$

where the summation runs on all partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ of n and χ_λ is the corresponding irreducible character of the symmetric group S_n . The explicit form of the multiplicities $m_\lambda(R)$ is known for few algebras only, among them the Grassmann algebra E (Krakowski and Regev [KR], Olsson and Regev [OR]), the 2×2 matrix algebra $M_2(K)$ (Formanek [F1] and Drensky [D2]), the algebra $U_2(K)$ of the 2×2 upper triangular matrices (Mishchenko, Regev and Zaicev [MRZ], based on the approach of Berele and Regev [BR1], see also [D6]), the tensor square $E \otimes E$ of the Grassmann algebra (Popov [P2], Carini and Di Vincenzo [CDV]), the algebra $U_2(E)$ of 2×2 upper triangular matrices with Grassmann entries (Centrone [Ce]).

The n -th cocharacter $\chi_n(R)$ is equal to the character of the representation of S_n acting on the vector subspace $P_n \subset K\langle X \rangle$ of the multilinear polynomials of degree n modulo the polynomial identities of R . It is related with another important group

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action, namely the action of the general linear group $GL_d = GL_d(K)$ on the d -generated free subalgebra $K\langle X_d \rangle = K\langle x_1, \dots, x_d \rangle \subset K\langle X \rangle$ modulo the polynomial identities in d variables of R . The algebra

$$F_d(R) = K\langle X_d \rangle / (K\langle X_d \rangle \cap T(R)) = K\langle x_1, \dots, x_d \rangle / (K\langle x_1, \dots, x_d \rangle \cap T(R))$$

is called the relatively free algebra of rank d in the variety of algebras $\text{var}(R)$ generated by the algebra R . It is \mathbb{Z}_d -graded with grading defined by

$$\deg(x_1) = (1, 0, \dots, 0), \deg(x_2) = (0, 1, \dots, 0), \dots, \deg(x_d) = (0, 0, \dots, 1).$$

The Hilbert series

$$H(F_d(R), T_d) = H(F_d(R), t_1, \dots, t_d) = \sum_{n_i \geq 0} \dim(F_d^{(n_1, \dots, n_d)}(R)) t_1^{n_1} \dots t_d^{n_d}$$

of $F_d(R)$, where $F_d^{(n_1, \dots, n_d)}(R)$ is the homogeneous component of degree (n_1, \dots, n_d) of $F_d(R)$, is a symmetric function which plays the role of the character of the corresponding GL_d -representation. The Schur functions $S_\lambda(T_d) = S_\lambda(t_1, \dots, t_d)$ are the characters of the irreducible GL_d -submodules $W_d(\lambda)$ of $F_d(R)$ and

$$H(F_d(R), T_d) = \sum_{\lambda} m_{\lambda}(R) S_{\lambda}(T), \quad \lambda = (\lambda_1, \dots, \lambda_d).$$

By a result of Berele [B1] and Drensky [D1, D2], the multiplicities $m_{\lambda}(R)$ are the same as in the cocharacter sequence $\chi_n(R)$, $n = 0, 1, 2, \dots$. Hence, in principle, if we know the Hilbert series $H(F_d(R), T_d)$, we can find the multiplicities $m_{\lambda}(R)$ in $\chi_n(R)$ for those λ which are partitions in not more than d parts. When R is a finite dimensional algebra, the multiplicities $m_{\lambda}(R)$ are equal to zero for partitions $\lambda = (\lambda_1, \dots, \lambda_d)$, $\lambda_d \neq 0$, for $d > \dim(R)$, see Regev [R2]. Hence all $m_{\lambda}(R)$ can be recovered from $H(F_d(R), T_d)$ for d sufficiently large. Following the idea of Drensky and Genov [DG1] we consider the multiplicity series of R

$$M(R; T_d) = M(R; t_1, \dots, t_d) = \sum_{\lambda} m_{\lambda}(R) T_d^{\lambda} = \sum_{\lambda} m_{\lambda}(R) t_1^{\lambda_1} \dots t_d^{\lambda_d}.$$

This is the generating function of the cocharacter sequence of R which corresponds to the multiplicities $m_{\lambda}(R)$ when λ is a partition in $\leq d$ parts. Then, if we know the Hilbert series $H(F_d(R), T_d)$, the problem is to compute the multiplicity series $M(R; T_d)$ and to find its coefficients. This problem was solved in [DG2] for rational symmetric functions of special kind and in two variables. Berele [B2] suggested another approach involving the so called nice rational functions which allowed, see Berele and Regev [BR2] and Berele [B3], to solve for unital algebras the conjecture of Regev about the precise asymptotics of the growth of the codimension sequence of PI-algebras. But the results of [B2, BR2, B3] do not give explicit algorithms to find the multiplicities of the irreducible characters. One can apply classical algorithms to find the multiplicity series when the Hilbert series of the relatively free algebra is known. These algorithms follow from the method of Elliott [E], improved by MacMahon [MM] in his “ Ω -Calculus” or Partition Analysis, with further improvements and computer realizations, see Andrews, Paule and Riese [APR] and Xin [X]. See also the series of twelve papers on MacMahon’s partition analysis by Andrews, alone or jointly with Paule, Riese and Strehl (I – [A], \dots , XII – [AP]).

Formanek [F2] expressed the Hilbert series of the product of two T-ideals in terms of the Hilbert series of the factors. Berele and Regev [BR1] translated this result in the language of cocharacters. If $\chi_n(R_1)$ and $\chi_n(R_2)$ are, respectively, the

cocharacter sequences of the algebras R_1 and R_2 , then the cocharacter sequence of the T-ideal $T(R) = T(R_1)T(R_2)$ is

$$\chi_n(R) = \chi_n(R_1) + \chi_n(R_2) + \chi_{(1)} \hat{\otimes} \sum_{j=0}^{n-1} \chi_j(R_1) \hat{\otimes} \chi_{n-j-1}(R_2) - \sum_{j=0}^n \chi_j(R_1) \hat{\otimes} \chi_{n-j}(R_2),$$

where $\hat{\otimes}$ denotes the “outer” tensor product of characters. For irreducible characters it corresponds to the Littlewood-Richardson rule for products of Schur functions:

$$\chi_\lambda \hat{\otimes} \chi_\mu = \sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda\mu}^\nu \chi_\nu,$$

where

$$S_\lambda(T_d) S_\mu(T_d) = \sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda\mu}^\nu S_\nu(T_d),$$

$$\lambda = (\lambda_1, \dots, \lambda_p), \quad \mu = (\mu_1, \dots, \mu_q), \quad d \geq p + q.$$

In the present paper we study the cocharacter sequence of the algebra $U_k = U_k(K)$ of $k \times k$ upper triangular matrices. The algebra U_k is one of the central objects in the theory of PI-algebras satisfying a nonmatrix polynomial identity (i.e., an identity which does not hold for the 2×2 matrix algebra $M_2(K)$). Latyshev [L1] proved that every finitely generated PI-algebra with a nonmatrix identity satisfies the identities of U_k for a suitable k . Hence the polynomial identities of U_k may serve as a measure of the complexity of the polynomial identities of finitely generated algebras with nonmatrix identity in the same way as the polynomial identities of the $k \times k$ matrix algebra $M_k(K)$ measure the complexity of the identities of arbitrary PI-algebras, see [L3].

Yu. Maltsev [Ma] showed that the polynomial identities of U_k follow from the identity

$$[x_1, x_2] \cdots [x_{2k-1}, x_{2k}] = 0,$$

where $[x, y] = xy - yx$ is the commutator of x and y . This means that $T(U_k) = C^k$, where

$$C = T(K) = K\langle X \rangle [K\langle X \rangle, K\langle X \rangle] K\langle X \rangle$$

is the commutator ideal of $K\langle X \rangle$. Using purely combinatorial methods (the technique of partial ordered sets due to Higman [Hi] and Cohen [C]) Genov [G1, G2] and Latyshev [L2] proved that every algebra satisfying the identities of U_k has a finite basis of its polynomial identities. Later this result was generalized by Latyshev [L4] and Popov [P1] for PI-algebras satisfying the identity

$$[x_1, x_2, x_3] \cdots [x_{3k-2}, x_{3k-1}, x_{3k}] = 0$$

which generates the T-ideal $T(U_k(E)) = T^k(E)$ of the algebra $U_k(E)$ of $k \times k$ upper triangular matrices with entries from the Grassmann algebra E . For long time, until Kemer developed his structure theory of T-ideals and solved the Specht problem (i.e., finite basis problem) for arbitrary PI-algebras ([K1, K2], see also his book [K3] for an account of the theory), the theorems of Genov, Latyshev and Popov [G1, G2, L2, L4, P1] covered all known examples of classes of PI-algebras with the finite basis property.

Using methods of commutative algebra Krasilnikov [Kr] established that every Lie algebra satisfying the Lie polynomial identities of U_k has a finite basis of its Lie identities. His approach works also for associative algebras and gives a simple

proof of the result of Genov [G1, G2] and Latyshev [L2]. Drensky [D3], using the method of Krasilnikov [Kr] showed that the Hilbert series of every finitely generated relatively free algebra with nonmatrix polynomial identity is a rational function. Later this fact was generalized by Belov [Be] (see [KBR] for detailed exposition) for arbitrary finitely generated relatively free algebras, using the theory of Kemer [K3].

The T-ideals of the algebras $U_k(K)$ and $U_k(E)$ have another interesting property [D4]. They are examples of maximal T-ideals of a given exponent of the codimension sequences (and the corresponding varieties of algebras are minimal varieties of this exponent). Later, Giambruno and Zaicev [GZ1, GZ2] used the theory of Kemer [K3] combined with methods of representation theory of the symmetric groups and proved the conjecture of [D4] that the only maximal T-ideals of a given exponent are the products $T(R_1) \cdots T(R_k)$, where $T(R_i)$ are T-prime T-ideals from the structure theory of T-ideals developed by Kemer [K1].

In this paper we use the result of Formanek [F2] and calculate the Hilbert series $H(F_d(U_k), T_d)$ for any k and d :

$$\begin{aligned} H(F_d(U_k), T_d) &= \frac{1}{t_1 + \cdots + t_d - 1} \left(\left(1 + (t_1 + \cdots + t_d - 1) \prod_{i=1}^d \frac{1}{1 - t_i} \right)^k - 1 \right) \\ &= \sum_{j=1}^k \binom{k}{j} \left(\prod_{i=1}^d \frac{1}{1 - t_i} \right)^j (t_1 + \cdots + t_d - 1)^{j-1}. \end{aligned}$$

Hence $H(F_d(U_k), T_d)$ is a linear combination of expressions of the form

$$\left(\prod_{i=1}^d \frac{1}{1 - t_i} \right)^p (t_1 + \cdots + t_d)^q, \quad 0 \leq q < p \leq k.$$

If two symmetric functions $f(T_d)$ and $g(T_d)$ are related by

$$f(T_d) = g(T_d) \prod_{i=1}^d \frac{1}{1 - t_i},$$

then $f(T_d)$ is Young-derived from $g(T_d)$ and the decomposition of $f(T_d)$ as a series of Schur functions can be obtained from the decomposition of $g(T_d)$ using the Young rule.

Using the decomposition

$$(t_1 + \cdots + t_d)^q = \sum_{\lambda \vdash q} d_\lambda S_\lambda(T_d),$$

where d_λ is the degree of the irreducible S_q -character χ_λ , it is sufficient to apply the Young rule up to k times on the Schur functions $S_\lambda(T_d)$ for all partitions λ of $q \leq k - 1$. It has turned out that if $m_\lambda(U_k(K))$ is different from zero, then $\lambda = (\lambda_1, \dots, \lambda_{2k-1})$ is a partition in not more than $2k - 1$ parts and $\bar{\lambda} = (\lambda_{k+1}, \dots, \lambda_{2k-1})$ is a partition of $i \leq k - 1$.

We have found the exact value of $m_\lambda(U_k(K))$ for the “largest” partitions with $\bar{\lambda}$ a partition of $k - 1$. We use the fact that in the decomposition

$$\prod_{i=1}^k \frac{1}{(1 - t_i)^k} = \sum_{\mu} n_\mu S_\mu(T_k), \quad \mu = (\mu_1, \dots, \mu_k),$$

the coefficient n_μ of $S_\mu(T_k)$ is equal to the dimension of the irreducible GL_k -module $W_k(\mu)$ corresponding to the partition μ . We obtain for $\bar{\lambda} \vdash k-1$,

$$m_\lambda(U_k) = d_{\bar{\lambda}} \dim(W_k(\lambda_1, \dots, \lambda_k)),$$

where $d_{\bar{\lambda}}$ is the degree of the S_{k-1} -character $\chi_{\bar{\lambda}}$ and

$$\dim(W_k(\lambda_1, \dots, \lambda_k)) = S_\lambda(\underbrace{1, \dots, 1}_k) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

There is an easy algorithm from [DG1] with input the multiplicity series of a symmetric function and output the multiplicity series of its Young-derived. Applying it, we have found the explicit form of the multiplicity series of $H(F_d(U_k), T_d)$ and the multiplicities $m_\lambda(U_k)$ for the first several k and any λ .

The main results of this paper have been announced without proofs in [BD].

1. PRELIMINARIES

We fix a positive integer d and consider the algebra

$$\mathbb{C}[[T_d]] = \mathbb{C}[[t_1, \dots, t_d]]$$

of formal power series in d commuting variables. As usually, if

$$f(T_d) = \frac{p(T_d)}{q(T_d)} \in \mathbb{C}(T_d), \quad p(T_d), q(T_d) \in \mathbb{C}[T_d], q(T_d) \neq 0,$$

is a rational function and $g(T_d) \in \mathbb{C}[[T_d]]$ is such that $p(T_d) = q(T_d)g(T_d)$ in $\mathbb{C}[[T_d]]$, we shall identify $f(T_d)$ and $g(T_d)$. Let $\mathbb{C}[[T_d]]^{S_d}$ be the subalgebra of symmetric functions. Every symmetric function $f(T_d) \in \mathbb{C}[[T_d]]^{S_d}$ can be presented in the form

$$f(T_d) = \sum_{\lambda} m_{\lambda} S_{\lambda}(T_d), \quad m_{\lambda} \in \mathbb{C}, \lambda = (\lambda_1, \dots, \lambda_d),$$

where $S_{\lambda}(T_d)$ is the Schur function related to λ . For details on the theory of Schur functions see the book by Macdonald [Mc]. As usually, we shall omit the zeros in the partitions and shall identify $(\lambda_1, \dots, \lambda_i, 0, \dots, 0)$ and $(\lambda_1, \dots, \lambda_i)$. There are several ways to define Schur functions. The most convenient for our purposes is to define them as fractions of Vandermonde type determinants:

$$S_{\lambda}(T_d) = \frac{V(\lambda + \delta)}{V(\delta)},$$

where $\delta = (d-1, \dots, 2, 1)$ and for $\mu = (\mu_1, \dots, \mu_d)$

$$V(\mu, T_d) = \begin{vmatrix} t_1^{\mu_1} & t_2^{\mu_1} & \cdots & t_d^{\mu_1} \\ t_1^{\mu_2} & t_2^{\mu_2} & \cdots & t_d^{\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{\mu_d} & t_2^{\mu_d} & \cdots & t_d^{\mu_d} \end{vmatrix}.$$

We shall denote by $[\lambda]$ the Ferrers (or Young) diagram of λ . Recall that the λ -tableau D_{λ} is semistandard if its entries do not decrease in rows reading from left to right, and increase strictly in columns reading from top to bottom. The tableau

is of contents $\alpha = \alpha(D_\lambda) = (\alpha_1, \dots, \alpha_d)$ if each integer $i = 1, \dots, d$ appears in the tableau exactly α_i times.

1	1	2	2	2	3	4
2	2	3	4	4		
3	6	6	7			
5	7					

A semistandard $(7, 5, 4, 2)$ -tableau
of contents $(2, 5, 3, 3, 1, 2, 2)$.

Another presentation of Schur functions is given in terms of semistandard Young tableaux:

$$S_\lambda(T_d) = \sum T^{\alpha(D_\lambda)},$$

where the summation runs on all semistandard λ -tableaux.

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$$S_{(2,1)}(T_3) = \sum_{i \neq j} t_i^2 t_j + 2t_1 t_2 t_3.$$

It is well known that Schur functions play the role of characters of irreducible polynomial representations of GL_d . If $W_d(\lambda)$ is the irreducible GL_d -module labeled by the partition λ , then

$$\dim(W_d(\lambda)) = S_\lambda(\underbrace{1, \dots, 1}_{d \text{ times}}) = \prod_{1 \leq i < j \leq d} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

We associate with

$$f(T_d) = \sum_{\lambda} m_{\lambda} S_{\lambda}(T_d)$$

its multiplicity series

$$M(f; T_d) = \sum_{\lambda} m_{\lambda} T_d^{\lambda} = \sum_{\lambda} m_{\lambda} t_1^{\lambda_1} \dots t_d^{\lambda_d} \in \mathbb{C}[[T_d]].$$

It is also convenient to consider the subalgebra $\mathbb{C}[[V_d]] \subset \mathbb{C}[[T_d]]$ of the formal power series in the new set of variables $V_d = \{v_1, \dots, v_d\}$, where

$$v_1 = t_1, v_2 = t_1 t_2, \dots, v_d = t_1 \dots t_d.$$

Then the multiplicity series $M(f; T_d)$ can be written as

$$M'(f; V_d) = \sum_{\lambda} m_{\lambda} v_1^{\lambda_1 - \lambda_2} \dots v_{d-1}^{\lambda_{d-1} - \lambda_d} v_d^{\lambda_d} \in \mathbb{C}[[V_d]].$$

We also call $M'(f; V_d)$ the multiplicity series of f . The advantage of the mapping $M' : \mathbb{C}[[T_d]]^{S_d} \rightarrow \mathbb{C}[[V_d]]$ defined by $M' : f(T_d) \rightarrow M'(f; V_d)$ is that it is a bijection.

For a PI-algebra R we define the multiplicity series of R

$$M(R; T_d) = M(R; t_1, \dots, t_d) = \sum_{\lambda} m_{\lambda}(R) T_d^{\lambda} = \sum_{\lambda} m_{\lambda}(R) t_1^{\lambda_1} \dots t_d^{\lambda_d}.$$

Similarly we define the series $M'(R; V_d)$.

Lemma 1. (Berele [B2]) *The functions $f(T_d) \in \mathbb{C}[[T_d]]^{S_d}$ and $M(f; T_d)$ are related by the following equality. If*

$$f(T_d) \prod_{i < j} (t_i - t_j) = \sum_{p_i \geq 0} b(p_1, \dots, p_d) t_1^{p_1} \cdots t_d^{p_d}, \quad b(p_1, \dots, p_d) \in \mathbb{C},$$

then

$$M(f; T_d) = \frac{1}{t_1^{d-1} \cdots t_{d-2}^2 t_{d-1}} \sum_{p_i > p_{i+1}} b(p_1, \dots, p_d) t_1^{p_1} \cdots t_d^{p_d},$$

where the summation is on all $p = (p_1, \dots, p_d)$ such that $p_1 > p_2 > \cdots > p_d$.

Remark 2. In the general case, it is difficult to find $M(f; T_d)$ if we know $f(T_d)$. But it is very easy to check whether the formal power series

$$h(T_d) = \sum h(q_1, \dots, q_d) t_1^{q_1} \cdots t_d^{q_d}, \quad q_1 \geq \cdots \geq q_d,$$

is equal to the multiplicity series $M(f; T_d)$ of $f(T_d)$ because $h(T_d) = M(f; T_d)$ if and only if

$$f(T_d) \prod_{i < j} (t_i - t_j) = \sum_{\sigma \in S_d} \text{sign}(\sigma) t_{\sigma(1)}^{d-1} t_{\sigma(2)}^{d-2} \cdots t_{\sigma(d-1)} h(t_{\sigma(1)}, \dots, t_{\sigma(d)}).$$

This equation can be used to verify the computational results on multiplicities.

The Young rule in representation theory of the symmetric groups describes in the language of Young diagrams the induced S_{m+n} -character of the $S_m \times S_n$ -character $\chi_{(m)} \otimes \chi_\mu$, $\mu \vdash n$ (which is equal to the outer tensor product $\chi_{(m)} \hat{\otimes} \chi_\mu$). In the special case $m = 1$ it is equivalent to the Branching theorem for the induced S_{n+1} -character of χ_μ , $\mu \vdash n$. Translated in the language of Schur functions the Young rule is stated as

$$S_{(m)}(T_d) S_\mu(T_d) = \sum_{\lambda} S_\lambda(T_d),$$

where the summation is over all partitions λ such that the skew diagram $[\lambda/\mu]$ is a horizontal strip of size m , i.e.,

$$\lambda_1 + \cdots + \lambda_d = \mu_1 + \cdots + \mu_d + m,$$

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_d \geq \mu_d.$$

Regev [R3] introduced the notion of Young-derived sequences of S_n -characters. The sequence ζ_n , $n = 0, 1, 2, \dots$, is Young-derived if it is obtained from another sequence of S_k -characters ξ_k , $k = 0, 1, 2, \dots$, by applying the Young rule:

$$\zeta_n = \sum_{k=0}^n \xi_{(n-k)} \hat{\otimes} \xi_k, \quad n = 0, 1, 2, \dots$$

In terms of symmetric functions this means that the symmetric function

$$f(T_d) = \sum_{\lambda} m_{\lambda} S_{\lambda}(T_d)$$

is Young-derived from

$$g(T_d) = \sum_{\lambda} p_{\lambda} S_{\lambda}(T_d)$$

if and only if the multiplicities m_{λ} and p_{μ} are related with the condition

$$m_{(\lambda_1, \dots, \lambda_d)} = \sum p_{(\mu_1, \dots, \mu_d)}, \quad \lambda_1 \geq \mu_1 \geq \cdots \geq \lambda_d \geq \mu_d.$$

The well known equality

$$\prod_{i=1}^d \frac{1}{1-t_i} = \sum_{m \geq 0} S_{(m)}(T_d)$$

gives that this is equivalent to the equality

$$f(T_d) = g(T_d) \prod_{i=1}^d \frac{1}{1-t_i}.$$

Let Y be the linear operator in $\mathbb{C}[[V_d]] \subset \mathbb{C}[[T_d]]$ which sends the multiplicity series of the symmetric function $g(T_d)$ to the multiplicity series of its Young-derived $f(T_d)$:

$$Y(M(g); T_d) = M(f; T_d) = M\left(g(T_d) \prod_{i=1}^d \frac{1}{1-t_i}; T_d\right).$$

The following proposition describes the multiple action of Y on 1. We believe that it is folklorically known although we were not able to find references.

Proposition 3. *For $d \geq k \geq 1$ the following decomposition holds*

$$\prod_{i=1}^d \frac{1}{(1-t_i)^k} = \sum_{\mu} n_{\mu} S_{\mu}(T_d),$$

where the summation is on all partitions $\mu = (\mu_1, \dots, \mu_k)$ and

$$n_{\mu} = S_{\mu}(\underbrace{1, \dots, 1}_{k \text{ times}}) = \dim(W_k(\mu)).$$

Equivalently,

$$Y^k(1) = \sum_{\mu} \dim(W_k(\mu)) T_k^{\mu}, \quad \mu = (\mu_1, \dots, \mu_k), \quad k \geq 1.$$

Proof. Clearly

$$\prod_{i=1}^d \frac{1}{(1-t_i)^k} = \sum S_{(m_1)}(T_d) \cdots S_{(m_k)}(T_d),$$

where the summation is on all k -tuples of nonnegative integers (m_1, \dots, m_k) . By the Young rule

$$S_{(m_1)}(T_d) S_{(m_2)}(T_d) = \sum S_{\pi}(T_d),$$

where the sum is on all partitions $\pi = (\pi_1, \pi_2) \vdash m_1 + m_2$ such that $\pi_1 \geq m_1$ and the skew diagram $[\pi/(m_1)]$ is a horizontal strip. We fill in the entries of $[(m_1)]$ and $[(m_2)]$ with 1's and 2's, respectively. Then we fill in with 1's and 2's the boxes of $[\pi]$ corresponding to the boxes of $[(m_1)]$ and $[(m_2)]$, respectively:

$$\boxed{1 \cdots 1} \otimes \boxed{2 \cdots 2} = \sum \boxed{\begin{array}{cccccc} 1 & \cdots & 1 & 1 & \cdots & 1 \\ 2 & \cdots & 2 & & & \end{array}}$$

As a result, we obtain a bijection between the summands $S_{\pi}(T_d)$ in the decomposition of the product $S_{(m_1)}(T_d) S_{(m_2)}(T_d)$ and the semistandard tableaux of content (m_1, m_2) . In the next step, the product of three Schur functions has the form

$$S_{(m_1)}(T_d) S_{(m_2)}(T_d) S_{(m_3)}(T_d) = \sum S_{\rho}(T_d),$$

where the sum is on all partitions $\rho = (\rho_1, \rho_2, \rho_3) \vdash m_1 + m_2 + m_3$ which contain a partition $\pi = (\pi_1, \pi_2) \vdash m_1 + m_2$ such that the skew diagrams $[\pi/(m_1)]$ and $[\rho/\pi]$ are horizontal strips. The Schur function $S_\rho(T_d)$ participates in the sum as many times as the possible ways to choose the partition π . Hence $S_\rho(T_d)$ appears in the sum with its multiplicity in the decomposition of $S_{(m_1)}(T_d)S_{(m_2)}(T_d)S_{(m_3)}(T_d)$. Again, filling in the entries of $[(m_1)]$, $[(m_2)]$ and $[(m_3)]$ with 1's, 2's and 3's, respectively, we obtain a bijection between the summands $S_\rho(T_d)$ of $S_{(m_1)}(T_d)S_{(m_2)}(T_d)S_{(m_3)}(T_d)$ and the semistandard tableaux of content (m_1, m_2, m_3) :

$$\boxed{1 \dots 1} \otimes \boxed{2 \dots 2} \otimes \boxed{3 \dots 3} = \sum \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & \dots & 1 & 1 & \dots & 1 & 1 & \dots & 2 & 2 & \dots & 2 & 3 & \dots & 3 \\ \hline 2 & \dots & 2 & 2 & \dots & 2 & 3 & \dots & 3 & & & & & & \\ \hline 3 & \dots & 3 & & & & & & & & & & & & \\ \hline \end{array}$$

This bijection preserves the shape of the partitions and $S_\rho(T_d)$ is mapped to a ρ -tableau. Continuing in this way we obtain a bijection between the summands $S_\mu(T_d)$ in the decomposition of the product $S_{(m_1)}(T_d) \cdots S_{(m_k)}(T_d)$ and the semistandard tableaux of content (m_1, \dots, m_k) . This bijection counts the multiplicity of $S_\mu(T_d)$ and preserves the shape of μ . Hence the multiplicity of $S_\mu(T_d)$ in the decomposition of $\prod_{i=1}^d 1/(1-t_i)^k$ is equal to the number of semistandard μ -tableaux which is equal to $S_\mu(1, \dots, 1)$ and to the dimension of the GL_k -module $W_k(\mu)$. The equivalence of both statements of the proposition is obvious. \square

Remark 4. The multiplicities in the decomposition of $\prod 1/(1-t_i)^k$ appear naturally in classical invariant theory. For example, see e.g. De Concini, Eisenbud and Procesi [DEP], if W is a direct sum of irreducible GL_d -modules $W_d(\lambda)$, the vector subspace W^{UT_d} of the invariants of the subgroup $UT_d = UT_d(K)$ of all upper unitriangular matrices of GL_d is spanned by elements $w_\lambda \in W_d(\lambda)$, where up to a multiplicative constant the element w_λ in $W_d(\lambda)$ is the only element with the property

$$\text{diag}(\xi_1, \dots, \xi_d)(w_\lambda) = \Xi^\lambda w_\lambda = \xi_1^{\lambda_1} \cdots \xi_d^{\lambda_d} w_\lambda$$

and $\text{diag}(\xi_1, \dots, \xi_d) \in GL_d$ is the diagonal matrix with the elements ξ_1, \dots, ξ_d at the diagonal. The vector subspace $W^{SL_d} \subset W^{UT_d}$ of the invariants of $SL_d = SL_d(K)$ is spanned by the elements w_λ such that $\lambda = (\lambda_1, \dots, \lambda_1)$ is a partition in d equal parts. The action of the diagonal subgroup of GL_d on the symmetric algebra $K[W_d(1)^{\oplus k}]$ of k copies of the canonical d -dimensional GL_d -module $W_d(1)$ induces a \mathbb{Z}^d -grading. The algebra $K[W_d(1)^{\oplus k}]^{UT_d}$ of UT_d -invariants is a graded subalgebra of $K[W_d(1)^{\oplus k}]$. Its Hilbert series is

$$H(K[W_1^{\oplus k}]^{UT_d}, T_d) = \sum_{\mu} n_{\mu} T_d^{\mu},$$

where n_{μ} is the multiplicity of $S_{\mu}(T_d)$ in $\prod 1/(1-t_i)^k$, i.e. is equal to the corresponding multiplicity series:

$$H(K[W_1^{\oplus k}]^{UT_d}, T_d) = M \left(\prod_{i=1}^d \frac{1}{(1-t_i)^k}; T_d \right).$$

Similarly, the algebra $K[W_1^{\oplus k}]^{SL_d}$ of SL_d -invariants is \mathbb{Z} -graded and its Hilbert series is

$$H(K[W_1^{\oplus k}]^{SL_d}, t) = \sum_{\mu} n_{(\mu_1, \dots, \mu_1)} t^{\mu_1 d} = M' \left(\prod_{i=1}^d \frac{1}{(1-t_i)^k}; 0, \dots, 0, t^d \right),$$

where $M'(f(T_d); V_d)$ is the multiplicity series of the symmetric function $f(T_d)$ with respect to the variables $v_i = t_1 \cdots t_i$, $i = 1, \dots, d$.

In the general case there is an easy formula which translates Young-derived sequences in the language of multiplicity series.

Proposition 5. (Drensky and Genov [DG1]) *Let $f(T_d)$ be the Young-derived of the symmetric function $g(T_d)$. Then*

$$\begin{aligned} Y(M(g); T_d) &= M(f; T_d) = M \left(g(T_d) \prod_{i=1}^d \frac{1}{1-t_i}; T_d \right) \\ &= \prod_{i=1}^d \frac{1}{1-t_i} \sum (-t_2)^{\varepsilon_2} \cdots (-t_d)^{\varepsilon_d} M(g; t_1 t_2^{\varepsilon_2}, t_2^{1-\varepsilon_2} t_3^{\varepsilon_3} \cdots t_{d-1}^{1-\varepsilon_{d-1}} t_d^{\varepsilon_d}, t_d^{1-\varepsilon_d}), \end{aligned}$$

where the summation runs on all $\varepsilon_2, \dots, \varepsilon_d = 0, 1$.

Remark 6. Applied to $S_{\mu}(T_d)$ when μ is a partition in $\leq p$ parts the Young rule gives a sum of Schur functions $S_{\lambda}(T_d)$ for partitions λ in $\leq p+1$ parts. Hence, if the multiplicity series $M(g; T_d)$ of $g(T_d)$ does not depend on t_{p+1} , then the multiplicity series of its Young-derived $Y(M(g); T_d)$ does not depend on t_{p+2} .

The proof of the following lemma can be derived directly from the Young rule, compare with the computing of the Hilbert series of the free metabelian Lie algebra in [D5]. Instead, we shall apply Proposition 5.

Lemma 7. *The following equality holds:*

$$\sum_{n \geq 2} S_{(n-1,1)}(T_d) = 1 + (t_1 + \cdots + t_d - 1) \prod_{i=1}^d \frac{1}{1-t_i}.$$

Proof. Starting with $M(S_{(1)}; T_d) = t_1$, we calculate $Y(t_1; T_d)$. It is sufficient to work for $d = 2$.

$$\begin{aligned} M \left(\frac{S_{(1)}}{(1-t_1)(1-t_2)}; t_1, t_2 \right) &= Y(t_1; T_2) = \frac{t_1 - t_2(t_1 t_2)}{(1-t_1)(1-t_2)} = \frac{t_1 + t_1 t_2}{1-t_1} \\ &= \sum_{n \geq 0} t^n - 1 + \sum_{n \geq 2} t_1^{n-1} t_2 \\ &= M \left(\sum_{n \geq 0} S_{(n)} - 1 + \sum_{n \geq 2} S_{(n-1,1)}; t_1, t_2 \right). \end{aligned}$$

Since the answer is the same for all $d \geq 2$, this is equivalent to

$$S_{(1)}(T_d) \prod_{i=1}^d \frac{1}{1-t_i} = \prod_{i=1}^d \frac{1}{1-t_i} - 1 + \sum_{n \geq 2} S_{(n-1,1)}(T_d),$$

and this completes the proof because

$$S_{(1)}(T_d) = t_1 + \cdots + t_d.$$

□

The following proposition expresses the Hilbert series of the product of two T-ideals in terms of the Hilbert series of the factors, and gives the corresponding relation for the Hilbert series of the relatively free algebras.

Proposition 8. (Formanek [F2], see also Halpin [H]) *Let R_1, R_2 and R be PI-algebras such that $T(R) = T(R_1)T(R_2)$. Then*

$$\begin{aligned} H(K\langle X_d \rangle, T_d)H(K\langle X_d \rangle \cap T(R), T_d) &= H(K\langle X_d \rangle \cap T(R_1), T_d)H(K\langle X_d \rangle \cap T(R_2), T_d), \\ H(F_d(R), T_d) &= H(F_d(R_1), T_d) + H(F_d(R_2), T_d) \\ &\quad + (t_1 + \cdots + t_d - 1)H(F_d(R_1), T_d)H(F_d(R_2), T_d). \end{aligned}$$

Proof. The first equation is presented in [F2] and [H]. The second equation follows immediately from the first if we take into account that

$$H(K\langle X_d \rangle, T_d) = \frac{1}{1 - (t_1 + \cdots + t_d)}$$

and for any unital algebra A the isomorphism

$$F_d(A) \cong K\langle X_d \rangle / T(A)$$

implies

$$H(F_d(A), T_d) = H(K\langle X_d \rangle, T_d) - H(T(A), T_d).$$

□

Finally, there is an explicit description of the relatively free algebra $F_d(U_k)$ and its Hilbert series in terms of tensor products of irreducible GL_d -modules and products of Schur functions.

Proposition 9. (Drensky and Kasparian [DK], see also [D6]) (i) *Let $W_d(\lambda)$ be the irreducible GL_d -module corresponding to the partition λ . Then the following GL_d -module isomorphism holds:*

$$F_d(U_k) \cong \left(\bigoplus_{n \geq 0} W_d(n) \right) \otimes \left(\bigoplus_{r=0}^{k-1} \bigoplus_{p_i \geq 2} W_d(p_1 - 1, 1) \otimes \cdots \otimes W_d(p_r - 1, 1) \right).$$

(ii) *The Hilbert series of $F_d(U_k)$ has the form*

$$H(F_d(U_k), T_d) = \prod_{i=1}^d \frac{1}{1 - t_i} \sum_{r=0}^{k-1} \sum_{p_i \geq 2} S_{(p_1-1,1)}(T_d) \cdots S_{(p_r-1,1)}(T_d).$$

Proof. By a theorem of Drensky [D2] (or [D6, Theorems 4.3.12 and 12.5.4]) if A is any unital PI-algebra and $B_d(A) \subset F_d(A)$ is the vector space of the so called “proper” polynomials in the relatively free algebra $F_d(A)$, then the Hilbert series of $F_d(A)$ and $B_d(A)$ are related by

$$H(F_d(U_k), T_d) = H(K[X_d], T_d)H(B_d(U_k), T_d) = \prod_{i=1}^d \frac{1}{1 - t_i} H(B_d(U_k), T_d)$$

and the following GL_d -module isomorphism holds:

$$F_d(A) \cong K[X_d] \otimes B_d(A) \cong \left(\bigoplus_{n \geq 0} W_d(n) \right) \otimes B_d(A).$$

Now we apply the decomposition in [DK], see also [D6, Theorem 12.5.6],

$$B_d(U_k) \cong \bigoplus_{r=0}^{k-1} \bigoplus_{p_i \geq 2} W_d(p_1 - 1, 1) \otimes \cdots \otimes W_d(p_r - 1, 1)$$

and its counterpart in the language of Schur functions

$$H(B_d(U_k), T_d) = \sum_{r=0}^{k-1} \sum_{p_i \geq 2} S_{(p_1-1,1)}(T_d) \cdots S_{(p_r-1,1)}(T_d)$$

and complete the proof. \square

Remark 10. Proposition 9 expresses the fact that the basis of the vector space $F_d(U_k)$ consists of the polynomials in d variables

$$x_1^{a_1} \cdots x_d^{a_d} [x_{i_{11}}, x_{i_{12}}, \dots, x_{i_{1p_1}}] \cdots [x_{i_{r1}}, x_{i_{r2}}, \dots, x_{i_{rp_r}}],$$

where $p_j \geq 2$, $j = 1, \dots, r$, $r = 0, 1, \dots, k-1$, and the subscripts in the commutators satisfy $i_{j1} > i_{j2} \leq \cdots \leq i_{jp_j}$. The commutators are left normed, i.e.,

$$[v_1, v_2, \dots, v_{p-1}, v_p] = [[\dots [v_1, v_2], \dots, v_{p-1}], v_p].$$

For fixed j the commutators $[x_{i_{j1}}, x_{i_{j2}}, \dots, x_{i_{jp_j}}]$, $p_j \geq 2$, form also a basis of the commutator ideal L'_d/L''_d of the free metabelian Lie algebra L_d/L''_d , where L_d is the free Lie algebra of rank d . It is well known that the GL_d -module L'_d/L''_d is a direct sum of $W_d(p-1, 1)$ participating with multiplicity 1, $p \geq 2$, and this means that the Hilbert series of L'_d/L''_d is

$$H(L'_d/L''_d, T_d) = \sum_{p \geq 2} S_{(p-1,1)}(T_d).$$

Remark 11. The commutators $[x_{i_1}, x_{i_2}, \dots, x_{i_p}]$, $i_1 > i_2 \leq \cdots \leq i_p$, from Remark 10 generate the subalgebra of $K\langle X_d \rangle$ of the so called “proper” polynomials. By Gerritzen [Ge] it is a free algebra and these commutators form a free generating set (implicitly this is also in [DK]). This algebra is also the algebra of constants (i.e., the intersection of the kernels) of the formal partial derivatives $\partial/\partial x_i$, $i = 1, \dots, d$, as described by Falk [Fa]. (The freedom does not follow immediately from invariant theory of free algebras as developed by Lane [La] and Kharchenko [Kh]. The derivations $\partial/\partial x_i$ are locally nilpotent and the exponential automorphisms $\exp(\partial/\partial x_i)$ are affine automorphisms of $K\langle X_d \rangle$. But we cannot apply directly [La] and [Kh] because these automorphisms are not linear.) Specht [S] applied proper polynomials in the study of PI-algebras, see the book [D6] for further applications. To the best of our knowledge the papers by Specht [S] and Malcev [M] in 1950 are the first publications where representation theory of symmetric groups was involved in the study of PI-algebras. Later Regev in a series of brilliant papers, starting with the theorems for the exponential growth of the codimension sequence and the tensor product of PI-algebras [R1] in 1972, developed his powerful method for quantitative study of PI-algebras. The present paper is one of the many others which exploit the ideas of Regev.

2. MAIN RESULTS

We start the exposition of the main results of our paper with the Hilbert series of the relatively free algebras of the variety generated by the algebra of $k \times k$ upper triangular matrices.

Theorem 12. *The Hilbert series $H(F_d(U_k), T_d)$ of the algebra $F_d(U_k)$ is*

$$\begin{aligned} H(F_d(U_k), T_d) &= \frac{1}{t_1 + \cdots + t_d - 1} \left(\left(1 + (t_1 + \cdots + t_d - 1) \prod_{i=1}^d \frac{1}{1-t_i} \right)^k - 1 \right) \\ &= \sum_{j=1}^k \binom{k}{j} \left(\prod_{i=1}^d \frac{1}{1-t_i} \right)^j (t_1 + \cdots + t_d - 1)^{j-1}. \end{aligned}$$

Proof. One possible way to prove the theorem is the following. Since $T(U_k) = C^k$, where C is the commutator ideal of $K\langle X \rangle$, we have that $T(U_k) = T(U_1)T(U_{k-1})$. Proposition 8 gives the recurrent formula

$$\begin{aligned} H(F_d(U_k), T_d) &= H(F_d(U_1), T_d) + H(F_d(U_{k-1}), T_d) \\ &\quad + (t_1 + \cdots + t_d - 1)H(F_d(U_1), T_d)H(F_d(U_{k-1}), T_d). \end{aligned}$$

Since $F_d(U_1) = K\langle X_d \rangle / C \cong K[X_d]$, we start with the well known

$$H(F_d(U_1), T_d) = H(K[X_d], T_d) = \prod_{i=1}^d \frac{1}{1-t_i}$$

and complete the proof by induction. Instead, we provide a direct proof. By Proposition 9 (ii)

$$H(F_d(U_k), T_d) = \prod_{i=1}^d \frac{1}{1-t_i} \sum_{r=0}^{k-1} \left(\sum_{p_1 \geq 2} S_{(p_1-1,1)}(T_d) \right) \cdots \left(\sum_{p_r \geq 2} S_{(p_r-1,1)}(T_d) \right).$$

Applying Lemma 7 we obtain

$$\begin{aligned} H(F_d(U_k), T_d) &= \left(\prod_{i=1}^d \frac{1}{1-t_i} \right) \sum_{r=0}^{k-1} \left(1 + (t_1 + \cdots + t_d - 1) \prod_{i=1}^d \frac{1}{1-t_i} \right)^r \\ &= \left(\prod_{i=1}^d \frac{1}{1-t_i} \right) \frac{\left(1 + (t_1 + \cdots + t_d - 1) \prod_{i=1}^d \frac{1}{1-t_i} \right)^k - 1}{\left(1 + (t_1 + \cdots + t_d - 1) \prod_{i=1}^d \frac{1}{1-t_i} \right) - 1} \\ &= \left(\prod_{i=1}^d \frac{1}{1-t_i} \right) \frac{\left(1 + (t_1 + \cdots + t_d - 1) \prod_{i=1}^d \frac{1}{1-t_i} \right)^k - 1}{(t_1 + \cdots + t_d - 1) \prod_{i=1}^d \frac{1}{1-t_i}} \\ &= \frac{\left(1 + (t_1 + \cdots + t_d - 1) \prod_{i=1}^d \frac{1}{1-t_i} \right)^k - 1}{t_1 + \cdots + t_d - 1} \\ &= \sum_{j=1}^k \binom{k}{j} \left(\prod_{i=1}^d \frac{1}{1-t_i} \right)^j (t_1 + \cdots + t_d - 1)^{j-1}. \end{aligned}$$

□

Theorem 12 can be restated in the following way which, combined with Proposition 5, gives an algorithm to compute the multiplicity series of U_k .

Corollary 13. *Let Y be the linear operator in $\mathbb{C}[[V_d]] \subset \mathbb{C}[[T_d]]$ which sends the multiplicity series of the symmetric function $g(T_d)$ to the multiplicity series of its Young-derived:*

$$Y(M(g); T_d) = M\left(g(T_d) \prod_{i=1}^d \frac{1}{1-t_i}; T_d\right).$$

Then the multiplicity series of U_k is

$$M(U_k; T_d) = \sum_{j=1}^k \sum_{q=0}^{j-1} \sum_{\lambda \vdash q} (-1)^{j-q-1} \binom{k}{j} \binom{j-1}{q} d_\lambda Y^j(T_d^\lambda),$$

where d_λ is the degree of the irreducible S_n -character χ_λ and $T_d^\lambda = t_1^{\lambda_1} \cdots t_d^{\lambda_d}$ for $\lambda = (\lambda_1, \dots, \lambda_d)$.

Proof. Expanding the expression of $H(F_d(U_k), T_d)$ from Theorem 12 we obtain

$$\begin{aligned} H(F_d(U_k), T_d) &= \sum_{j=1}^k \binom{k}{j} \left(\prod_{i=1}^d \frac{1}{1-t_i} \right)^j (t_1 + \cdots + t_d - 1)^{j-1} \\ &= \sum_{j=1}^k \binom{k}{j} \left(\prod_{i=1}^d \frac{1}{1-t_i} \right)^j \sum_{q=0}^{j-1} (-1)^{j-q-1} \binom{j-1}{q} (t_1 + \cdots + t_d)^q. \end{aligned}$$

Now we use the well known equality

$$(t_1 + \cdots + t_d)^q = S_{(1)}^q(T_d) = \sum_{\lambda \vdash q} d_\lambda S_\lambda(T_d).$$

In the language of representation theory of the symmetric group applied to PI-algebras, it means that the S_q -character of the multilinear component P_q of degree q of the free algebra $K\langle X \rangle$ has the same decomposition as the character of the regular representation (i.e., of the character of the group algebra KS_q considered as a left S_q -module):

$$\chi_{S_q}(P_q) = \chi_{S_q}(KS_q) = \sum_{\lambda \vdash q} d_\lambda \chi_\lambda.$$

Hence

$$\begin{aligned} H(F_d(U_k), T_d) &= \sum_{j=1}^k \binom{k}{j} \left(\prod_{i=1}^d \frac{1}{1-t_i} \right)^j \sum_{q=0}^{j-1} (-1)^{j-q-1} \binom{j-1}{q} \sum_{\lambda \vdash q} d_\lambda S_\lambda(T_d) \\ &= \sum_{j=1}^k \sum_{q=0}^{j-1} \sum_{\lambda \vdash q} (-1)^{j-q-1} \binom{k}{j} \binom{j-1}{q} d_\lambda \left(\prod_{i=1}^d \frac{1}{1-t_i} \right)^j S_\lambda(T_d). \end{aligned}$$

This completes the proof because $M(S_\lambda(T_d); T_d) = T_d^\lambda$ and

$$M\left(\left(\prod_{i=1}^d \frac{1}{1-t_i}\right)^j S_\lambda(T_d); T_d\right) = Y^j(M(S_\lambda(T_d); T_d)) = Y^j(T_d^\lambda).$$

□

The following theorem describes the partitions λ with $m_\lambda(U_k) \neq 0$ and the explicit form of the multiplicities for the partitions of “maximal” shape.

Theorem 14. (i) If $m_\lambda(U_k) \neq 0$, then $\lambda = (\lambda_1, \dots, \lambda_{2k-1})$ is a partition in not more than $2k-1$ parts and $\bar{\lambda} = (\lambda_{k+1}, \dots, \lambda_{2k-1})$ is a partition of $i \leq k-1$.
(ii) If $\bar{\lambda}$ is a partition of $k-1$, then

$$m_\lambda(U_k) = d_{\bar{\lambda}} \dim(W_k(\lambda_1, \dots, \lambda_k)),$$

where $d_{\bar{\lambda}}$ is the degree of the S_{k-1} -character $\chi_{\bar{\lambda}}$ and

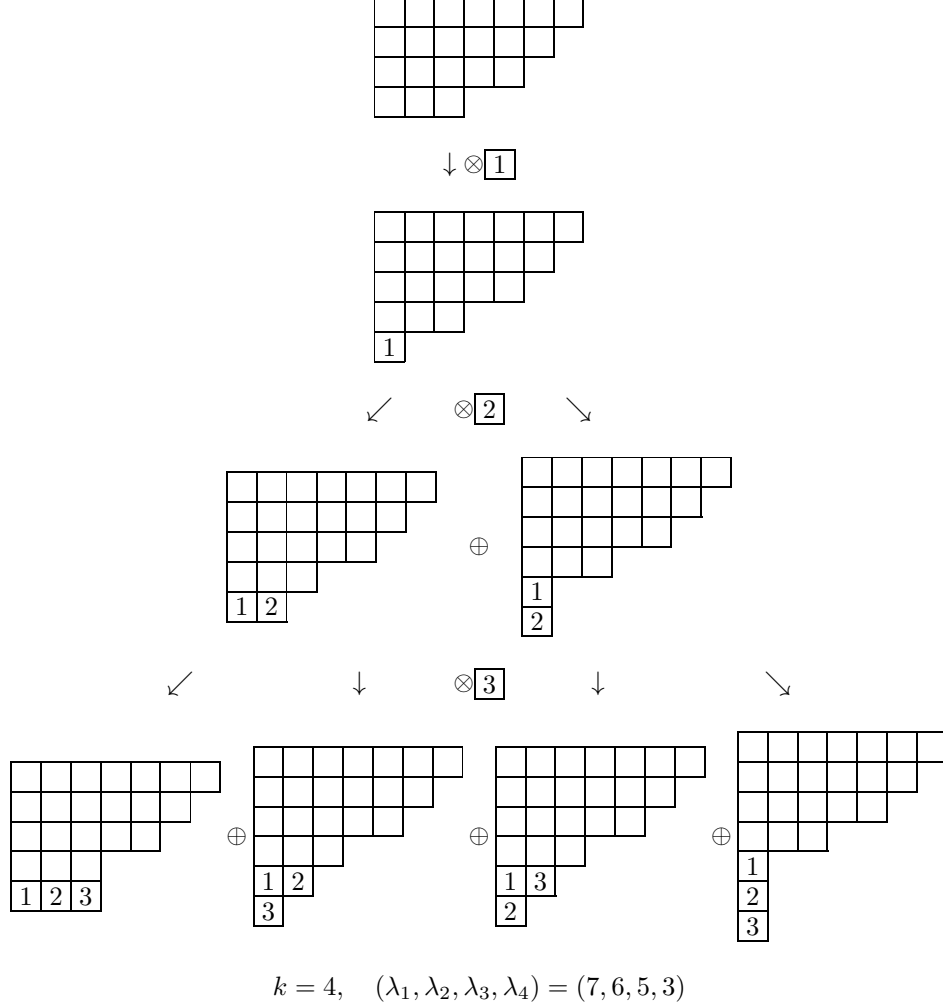
$$\dim(W_k(\lambda_1, \dots, \lambda_k)) = S_{\bar{\lambda}}(\underbrace{1, \dots, 1}_{k \text{ times}}) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Proof. (i) By Theorem 12 and in the spirit of Corollary 13 the nonzero multiplicities $m_\lambda(U_k)$ in the cocharacter sequence of U_k come from the decomposition as an infinite sum of Schur functions of

$$\left(\prod_{i=1}^d \frac{1}{1-t_i} \right)^j (t_1 + \dots + t_d)^q = (S_{(1)}(T_d))^q \left(\prod_{i=1}^d \frac{1}{1-t_i} \right)^j,$$

$j \leq k$, $q \leq k-1$. The Schur functions $S_\pi(T_d)$ participating in the product $(\prod_{i=1}^d 1/(1-t_i))^j$ are indexed by partitions π in $\leq j \leq k$ parts. By the Branching theorem the multiplication of $S_\pi(T_d)$ by $S_{(1)}(T_d)$ gives a sum of $S_\rho(T_d)$ where the diagrams $[\rho]$ are obtained from the diagram $[\pi]$ by adding a box. Clearly $[\rho]$ has not more than one box below the k -th row. Multiplying q times by $S_{(1)}(T_d)$ we add to the diagram $[\pi]$ not more than $q \leq k-1$ boxes below the k -th row. In this way, if $m_\lambda(U_k) \neq 0$, $\lambda = (\lambda_1, \dots, \lambda_n)$, then $\lambda_{k+1} + \dots + \lambda_n \leq k-1$ and obviously $\lambda_{2k} = 0$.

(ii) It follows from the proof of (i) that the multiplicity $m_\lambda(U_k)$ for $\bar{\lambda} \vdash k-1$ comes from the products of $S_\mu(T_d)$ and $(S_{(1)}(T_d))^{k-1}$, where $S_\mu(T_d)$ participates in the decomposition of $(\prod_{i=1}^d 1/(1-t_i))^k$, the diagram $[\mu] = [\mu_1, \dots, \mu_k]$ has exactly k rows, and all $k-1$ boxes added to $[\mu]$ to obtain $[\lambda]$ when multiplying $k-1$ times by $S_{(1)}(T_d)$ form the rows of $[\lambda]$ below the first k rows. Hence $\lambda_i = \mu_i$, $i = 1, \dots, k$. We fill in with i the box of the diagram corresponding to the i -th factor $S_{(1)}(T_d)$. As in the proof of Proposition 3, the $k-1$ boxes below the k -th row of the diagram $[\lambda]$ are filled in with the integers $1, \dots, k-1$ and form a standard $\bar{\lambda}$ -tableau. Again, there is a bijection between the standard $\bar{\lambda}$ -tableaux and the summands $S_{\bar{\lambda}}(T_d)$ obtained from a given $S_\mu(T_d) = S_{(\lambda_1, \dots, \lambda_k)}(T_d)$.



By Proposition 3 the multiplicity of $S_\mu(T_d)$ in the product $(\prod_{i=1}^d 1/(1-t_i))^k$ is equal to $\dim(W_k(\mu))$. This completes the proof because the number of the standard $\bar{\lambda}$ -tableaux is equal to the degree $d_{\bar{\lambda}}$ of the irreducible S_{k-1} -character $\chi_{\bar{\lambda}}$. \square

Using standard procedures of Maple only we have written a small program which computes the multiplicity series of U_k . It is more convenient to state the results of the computations for the difference $M'(U_k; V_d) - M'(U_{k-1}; V_d)$ than for $M'(U_k; V_d)$ or $M(U_k; T_d)$. (Recall that $M'(R; V_d) = M(R; T_d)$, where $v_i = t_1 \cdots t_i$, $i = 1, \dots, d$.) We give the results for the first several k . We write $M'(U_k; V)$ instead of $M'(U_k; V_d)$ assuming that the number of variables d is sufficiently large and has the property that $m_\lambda(U_k) = 0$ if $\lambda_{d+1} \neq 0$. The case $k = 1$ (when $U_1 = K$) is trivial and the case $k = 2$ is known (and can be computed applying the formula of Berele and Regev given in the introduction, see [MRZ], or directly by the Young rule and Proposition 9 (i)). We state the results for completeness.

Theorem 15. *The multiplicity series and the multiplicities of the cocharacter sequence of the algebra U_k of the $k \times k$ upper triangular matrices for $k = 1, 2$ are*

$$M'(U_1; V) = \frac{1}{1 - v_1}, \quad m_\lambda(U_1) = \begin{cases} 1, & \lambda = (\lambda_1) \\ 0, & \lambda_2 > 0; \end{cases}$$

$$M'(U_2; V) - M'(U_1; V) = \frac{v_2 + v_3}{(1 - v_1)^2(1 - v_2)},$$

$$m_\lambda(U_2) - m_\lambda(U_1) = \begin{cases} \lambda_1 - \lambda_2 + 1, & \lambda = (\lambda_1, \lambda_2), \lambda_2 > 0, \\ \lambda_1 - \lambda_2 + 1, & \lambda = (\lambda_1, \lambda_2, 1), \\ 0, & \text{for all other } \lambda. \end{cases}$$

The results for U_3 are the following.

Theorem 16. (i) *The difference of the multiplicity series of U_3 and U_2 is*

$$M'(U_3; V) - M'(U_2; V) = \left(\frac{v_5 + v_4^2 + 4v_4 + 4v_3}{1 - v_3} + v_2^2 \right) \frac{1 - v_1 v_2}{(1 - v_1)^3(1 - v_2)^3} \\ - \frac{(v_2^2 - v_1 - 3v_2 + 3)v_4 + (v_1 v_2^2 - v_1 v_2 + v_2^2 - v_1 - 4v_2 + 4)v_3}{(1 - v_1)^3(1 - v_2)^3};$$

(ii) *The explicit form of the corresponding multiplicities is*

$$m_\lambda(U_3) - m_\lambda(U_2) = \begin{cases} n_\lambda, & \lambda = (\lambda_1, \lambda_2, \lambda_3, 2), \\ n_\lambda, & \lambda = (\lambda_1, \lambda_2, \lambda_3, 1, 1), \\ 4n_\lambda - c_\lambda, & \lambda = (\lambda_1, \lambda_2, \lambda_3, 1), \\ 4n_\lambda - c_\lambda, & \lambda = (\lambda_1, \lambda_2, \lambda_3), \lambda_3 > 0, \\ \frac{1}{2}\lambda_1(\lambda_1 - \lambda_2 + 1)(\lambda_2 - 1), & \lambda_2 \geq 2, \\ 0, & \text{for all other } \lambda, \end{cases}$$

where

$$n_\lambda = \dim(W_3(\lambda_1, \lambda_2, \lambda_3)) = \frac{1}{2}(\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_1 - \lambda_3 + 2)$$

and the correction c_λ is

$$c_\lambda = \begin{cases} \frac{1}{2}(\lambda_1 + 2)(\lambda_1 - \lambda_2 + 1)(\lambda_2 + 1), & \lambda = (\lambda_1, \lambda_2, 1, 1), \\ \frac{1}{2}(\lambda_1 + 3)(\lambda_1 - \lambda_2 + 1)(\lambda_2 + 2), & \lambda = (\lambda_1, \lambda_2, 1), \\ 0, & \text{for all other } \lambda. \end{cases}$$

Proof. (i) We have evaluated the multiplicity series of U_3 applying the algorithm from Corollary 13. Instead, we may apply Remark 2: We want to show that the rational function given in the statement (i) of the theorem and depending on the five variables v_1, \dots, v_5 is equal to $M'(U_3; V) - M'(U_2; V)$. It is sufficient to check that the only symmetric function with this rational function as its multiplicity series is equal to $H(F_d(U_3), T_d) - H(F_d(U_2), T_d)$. This can be easily verified using the formula from Remark 2.

(ii) We have expanded into a power series the rational expression for $M'(U_3; V) - M'(U_2; V)$ given in part (i) of the theorem using the equalities

$$\frac{v_1^{a_1} v_2^{a_2}}{(1 - v_1)^3(1 - v_2)^3} = \sum_{n_1 \geq a_1} \sum_{n_2 \geq a_2} \binom{n_1 - a_1 + 2}{2} \binom{n_2 - a_2 + 2}{2} v_1^{n_1} v_2^{n_2},$$

$$\frac{v_1^{a_1} v_2^{a_2} v_3^{a_3}}{(1-v_1)^3(1-v_2)^3(1-v_3)} = \sum_{n_i \geq a_i} \binom{n_1 - a_1 + 2}{2} \binom{n_2 - a_2 + 2}{2} v_1^{n_1} v_2^{n_2} v_3^{n_3}.$$

Easy manipulations give the explicit expressions for $m_\lambda(U_3) - m_\lambda(U_2)$. In particular,

$$\begin{aligned} \frac{1 - v_1 v_2}{(1-v_1)^3(1-v_2)^3} &= \frac{1}{(1-v_1)^3(1-v_2)^2} + \frac{1}{(1-v_1)^2(1-v_2)^3} - \frac{1}{(1-v_1)^2(1-v_2)^2} \\ &= \sum_{n_i \geq 0} \left(\binom{n_1+2}{2} \binom{n_2+1}{1} + \binom{n_1+1}{1} \binom{n_2+2}{2} - \binom{n_1+1}{1} \binom{n_2+1}{1} \right) v_1^{n_1} v_2^{n_2} \\ &= \frac{1}{2} \sum_{n_i \geq 0} (n_1+1)(n_2+1)(n_1+n_2+2) v_1^{n_1} v_2^{n_2}. \end{aligned}$$

Clearly, the formulas for $\lambda = (\lambda_1, \lambda_2, \lambda_3, 2)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3, 1, 1)$ follow also from Theorem 14. \square

We have computed the multiplicity series $M(U_4; T)$ but the results are too technical to be stated here.

The colength sequence of a PI-algebra R is defined as the sequence of the number of irreducible characters, counting the multiplicities, in the cocharacter sequence of R :

$$cl_n(R) = \sum_{\lambda \vdash n} m_\lambda(R), \quad n = 0, 1, 2, \dots$$

If the algebra R is finite dimensional then the generating function of the colength sequence, the colength series of R , can be obtained immediately from the multiplicity series $M(R; T_d)$ for a sufficiently large d :

$$cl(R; t) = \sum_{n \geq 0} cl_n(R) t^n = M(R; \underbrace{t, \dots, t}_{d \text{ times}}).$$

Theorems 15 and 16 for the multiplicity series of U_k (together with the calculations for U_4) give:

Corollary 17.

$$\begin{aligned} cl(U_1; t) &= \frac{1}{1-t}; \\ cl(U_2; t) - cl(U_1; t) &= \frac{t^2}{(1-t)^3}; \\ cl(U_3; t) - cl(U_2; t) &= \frac{t^4(3+6t+4t^2-2t^3-t^4)}{(1-t)^3(1-t^2)^3}; \\ cl(U_4; t) - cl(U_3; t) &= \frac{t^6 p(t)}{(1-t)^4(1-t^2)^6}, \end{aligned}$$

$$p(t) = 11 + 45t + 63t^2 - t^3 - 42t^4 - 24t^5 + 16t^6 + 12t^7 - 3t^8 - t^9.$$

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INSTITUTE OF MATHEMATICS AND INFORMATICS, BULGARIAN ACADEMY OF SCIENCES, 1113
SOFIA, BULGARIA

E-mail address: `silvi@math.bas.bg`, `drensky@math.bas.bg`